Q1 Ma | Lösungen zu S. 15/4 a) $O_n$ b) $O_n$ c) $O_n$ d) $U_n$

4.

a) $O_n = \frac{1}{n}\cdot f(\frac{1}{n}) + \frac{1}{n}\cdot f(\frac{2}{n}) + \ldots + \frac{1}{n}\cdot f(\frac{n-1}{n}) + \frac{1}{n}\cdot f(\frac{n}{n})$

$= \frac{1}{n}\cdot(\frac{1}{n}+1) + \frac{1}{n}\cdot(\frac{2}{n}+1) + \ldots + \frac{1}{n}\cdot(\frac{n-1}{n}+1) + \frac{1}{n}\cdot(\frac{n}{n}+1)$

$= \frac{1}{n}\cdot\frac{1}{n} + \frac{1}{n}\cdot 1 + \frac{1}{n}\cdot\frac{2}{n} + \frac{1}{n}\cdot 1 + \ldots + \frac{1}{n}\cdot\frac{n-1}{n} + \frac{1}{n}\cdot 1 + \frac{1}{n}\cdot\frac{n}{n} + \frac{1}{n}\cdot 1$

$= \frac{1}{n}\cdot\frac{1}{n} + \frac{1}{n}\cdot\frac{2}{n} + \ldots + \frac{1}{n}\cdot\frac{n-1}{n} + \frac{1}{n}\cdot\frac{n}{n} + n\cdot\frac{1}{n}\cdot 1$

$= \frac{1}{n^2}\cdot(1+2+\ldots+(n-1)+n)+1$ → Summenformel anwenden

$= \frac{1}{n^2}\cdot\left(\frac{n\cdot(n+1)}{2}\right)+1$

$= \frac{1}{n^2}\cdot\left(\frac{1}{2}n^2+\frac{1}{2}n\right)+1$

$= \frac{1}{2}+\frac{1}{2}\cdot\frac{1}{n} + 1$

$\Rightarrow \lim\limits_{n\to\infty} O_n = \lim\limits_{n\to\infty}\left(\frac{1}{2}+\overbrace{\frac{1}{2}\cdot\frac{1}{n}}^{\to 0}+1\right) = \frac{1}{2}+1 = \underline{1{,}5}$

b) Wegen fallender Funktion muss zur Berechnung von $O_n$ das linke Balkenende in die Funktion eingesetzt werden:

$O_n = \frac{2}{n}\cdot f(0) + \frac{2}{n}\cdot f(\frac{2\cdot 1}{n}) + \frac{2}{n}\cdot f(\frac{2\cdot 2}{n}) + \ldots + \frac{2}{n}\cdot f(\frac{2\cdot(n-2)}{n}) + \frac{2}{n}\cdot f(\frac{2\cdot(n-1)}{n})$

$= \frac{2}{n}\cdot(2-0) + \frac{2}{n}\cdot(2-\frac{2\cdot 1}{n}) + \frac{2}{n}\cdot(2-\frac{2\cdot 2}{n}) + \ldots + \frac{2}{n}\cdot(2-\frac{2\cdot(n-2)}{n}) + \frac{2}{n}\cdot(2-\frac{2\cdot(n-1)}{n})$

$= \frac{2}{n}\cdot 2 + \frac{2}{n}\cdot 2 - \frac{2}{n}\cdot\frac{2\cdot 1}{n} + \frac{2}{n}\cdot 2 - \frac{2}{n}\cdot\frac{2\cdot 2}{n} \ldots + \frac{2}{n}\cdot 2 - \frac{2}{n}\cdot\frac{2\cdot(n-2)}{n} + \frac{2}{n}\cdot 2 - \frac{2}{n}\cdot\frac{2\cdot(n-1)}{n}$

$= n\cdot\frac{2}{n}\cdot 2 - \frac{4}{n^2}\cdot 1 - \frac{4}{n^2}\cdot 2 - \ldots - \frac{4}{n^2}\cdot(n-2) - \frac{4}{n^2}\cdot(n-1)$

$= 4 - \frac{4}{n^2}\cdot[1+2+\ldots+(n-2)+(n-1)]$ → Summenformel anwenden

$= 4 - \frac{4}{n^2}\cdot\left[\frac{(n-1)\cdot((n-1)+1)}{2}\right]$

$= 4 - \frac{4}{n^2}\cdot\left[\frac{1}{2}\cdot(n-1)\cdot n\right]$

$= 4 - \frac{4}{n^2}\cdot\left[\frac{1}{2}n^2 - \frac{1}{2}n\right]$

$= 4 - \frac{4}{2} + \frac{4}{2}\cdot\frac{1}{n}$

$\Rightarrow \lim\limits_{n\to\infty} O_n = \lim\limits_{n\to\infty}\left(4-2+2\cdot\frac{1}{n}\right) = 4-2 = \underline{\underline{2}}$





4. c)

$O_n = \frac{10}{n}\cdot f(\frac{10}{n}) + \frac{10}{n}\cdot f(\frac{20}{n}) + \frac{10}{n}\cdot f(\frac{30}{n}) + \ldots + \frac{10}{n}\cdot f(\frac{(n-1)\cdot 10}{n}) + \frac{10}{n}\cdot f(\frac{10n}{n})$

$= \frac{10}{n}\cdot\left(\frac{10}{n}\right)^2 + \frac{10}{n}\cdot\left(\frac{20}{n}\right)^2 + \frac{10}{n}\cdot\left(\frac{30}{n}\right)^2 + \ldots + \frac{10}{n}\cdot\left(\frac{(n-1)\cdot 10}{n}\right)^2 + \frac{10}{n}\cdot\left(\frac{10n}{n}\right)^2$

$= \frac{10}{n}\cdot\left(\frac{10\cdot 1}{n}\right)^2 + \frac{10}{n}\cdot\left(\frac{10\cdot 2}{n}\right)^2 + \frac{10}{n}\cdot\left(\frac{10\cdot 3}{n}\right)^2 + \ldots + \frac{10}{n}\cdot\left(\frac{10\cdot(n-1)}{n}\right)^2 + \frac{10}{n}\cdot\left(\frac{10\cdot n}{n}\right)^2$

$= \frac{10}{n}\cdot\frac{10^2\cdot 1^2}{n^2} + \frac{10}{n}\cdot\frac{10^2\cdot 2^2}{n^2} + \frac{10}{n}\cdot\frac{10^2\cdot 3^2}{n^2} + \ldots + \frac{10}{n}\cdot\frac{10^2\cdot(n-1)^2}{n^2} + \frac{10}{n}\cdot\frac{10^2\cdot n^2}{n^2}$

$= \frac{10\cdot 10^2}{n^3}\cdot\left[1^2+2^2+3^2+\ldots+(n-1)^2+n^2\right]$ → Summenformel anwenden

$= \frac{1000}{n^3}\cdot\left[\frac{1}{6}\cdot n\cdot(n+1)\cdot(2n+1)\right]$

$= \frac{1000}{n^3}\cdot\left[\frac{1}{3}\cdot n^3 + \frac{1}{2}\cdot n^2 + \frac{1}{6}\cdot n\right]$

$= \frac{1000}{3} + \frac{1000}{2}\cdot\frac{1}{n} + \frac{1000}{6}\cdot\frac{1}{n^2}$

$\Rightarrow \lim\limits_{n\to\infty} O_n = \frac{1000}{3} = 333{,}\overline{3}$

d) $U_n = \frac{1}{n}\cdot f(0) + \frac{1}{n}\cdot f(\frac{1}{n}) + \frac{1}{n}\cdot f(\frac{2}{n}) + \ldots + \frac{1}{n}\cdot f(\frac{n-1}{n})$

$= \frac{1}{n}\cdot(2\cdot(0)^2+0) + \frac{1}{n}\cdot(2\cdot(\frac{1}{n})^2+\frac{1}{n}) + \frac{1}{n}\cdot(2\cdot(\frac{2}{n})^2+\frac{2}{n}) + \ldots + \frac{1}{n}\cdot(2\cdot(\frac{n-1}{n})^2+\frac{n-1}{n})$

$= 0 + 2\cdot\frac{1}{n}\cdot(\frac{1}{n})^2 + \frac{1}{n}\cdot\frac{1}{n} + 2\cdot\frac{1}{n}\cdot(\frac{2}{n})^2 + \frac{1}{n}\cdot\frac{2}{n} + \ldots + 2\cdot\frac{1}{n}\cdot(\frac{n-1}{n})^2 + \frac{1}{n}\cdot\frac{n-1}{n}$

$= 2\cdot\frac{1}{n}\cdot(\frac{1}{n})^2 + 2\cdot\frac{1}{n}\cdot(\frac{2}{n})^2 + \ldots + 2\cdot\frac{1}{n}\cdot(\frac{n-1}{n})^2 + \frac{1}{n}\cdot\frac{1}{n} + \frac{1}{n}\cdot\frac{2}{n} + \ldots + \frac{1}{n}\cdot\frac{n-1}{n}$

$= 2\cdot\frac{1}{n}\cdot\frac{1}{n^2}\cdot\left[1^2+2^2+\ldots+(n-1)^2\right] + \frac{1}{n}\cdot\frac{1}{n}\cdot\left[1+2+\ldots+(n-1)\right]$ → Summenformel anwenden

$= \frac{2}{n^3}\cdot\left[\frac{1}{6}\cdot(n-1)\cdot n\cdot(2n-1)\right] + \frac{1}{n^2}\cdot\left[\frac{1}{2}\cdot(n-1)\cdot n\right]$

$= \frac{2}{n^3}\cdot\left[\frac{1}{3}n^3 - \frac{1}{2}\cdot n^2 + \frac{1}{6}\cdot n\right] + \frac{1}{n^2}\cdot\left[\frac{1}{2}n^2 - \frac{1}{2}\cdot n\right]$

$= \frac{2}{3} - \frac{1}{2}\cdot\frac{1}{n} + \frac{1}{6}\cdot\frac{1}{n^2} \quad + \frac{1}{2} - \frac{1}{2}\cdot\frac{1}{n}$

$= \frac{7}{6} - \frac{1}{n} + \frac{1}{6}\cdot\frac{1}{n^2}$

$\Rightarrow \lim\limits_{n\to\infty} U_n = \underline{\underline{\frac{7}{6}}}$